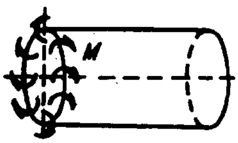

Fig. 3

$$n = \left(\frac{R}{eh}\right)^{1/2}, \qquad e = 2.718$$

For sufficiently large R / h the quantity q_0 is dependent on the thickness, and in the case of the effect of a ring loading on an infinite or semi-infinite shell will equal, respectively, $q_0 = 0.38$ and $q_0 = 0.18$, according to calculations utilizing (4.5) and (5.5).

When a system of moments distributed uniformly over the endface acts, the calculations yield $q_0 = 0.21$ according to (6.1).

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ENERGY CRITERION OF THE STABILITY OF ELASTIC BODIES WHICH DOES NOT REQUIRE THE DETERMINATION OF THE INITIAL STRESS-STRAIN STATE

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N. A. ALFUTOV and L. I. BALABUKH
(Moscow)

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It is shown that if the initial state of stress of a body is described by linear elasticity theory, then an energy criterion for neutral equilibrium can be formulated directly in terms of the external loading and the governing bifurcation of the displacements. To do this, besides the fundamental first order displacements, additional second order displacements on which external potential forces perform work during buckling, are introduced to describe the deflected equilibrium position of the body. These additional quadratic displacements are expressed in terms of the first order displacements. It therefore turns out that the stability problem of an elastic body can be solved without a preliminary determination of its initial state of stress. The result obtained can be considered as the foundation and extension of the energy stability criterion in the form of S. P. Timoshenko.

The energy stability criterion which does not require the initial stress determination is usually preferable when the initial state of stress is not uniform. Such a means of solution was utilized in [1] in examining the stability of plates.

1. Let us refer an elastic body of volume V to a rectangular x, y, z coordinate system; let $\overset{\circ}{p}_x$, $\overset{\circ}{p}_y$, $\overset{\circ}{p}_z$ and $F_x{}^\circ$, $F_y{}^\circ$, $F_z{}^\circ$, respectively, denote the components of the surface and volume potential loadings acting on the body. It is assumed that the initial stress-strain state of the body is described by equations of linear elasticity theory, i.e. the initial strains ε° are expressed linearly in terms of the initial displacements u°, v°, w°

$$\overset{\circ}{\varepsilon}_{xx} = \frac{\partial u^\circ}{\partial x}, \qquad \overset{\circ}{\varepsilon}_{xy} = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} \qquad (xyz,\ uvw) \tag{1.1}$$

Here and henceforth the symbol $(xyz,\ uvw)$ denotes cyclic commutation of the mentioned letters.

The initial stresses are related to the initial strains by Hooke's law

$$\overset{\circ}{\varepsilon}_{xx} = \frac{1}{E}\left(\overset{\circ}{\sigma}_{xx} - \mu\overset{\circ}{\sigma}_{yy} - \mu\overset{\circ}{\sigma}_{zz}\right), \qquad \overset{\circ}{\varepsilon}_{xy} = \frac{2(1+\mu)}{E}\overset{\circ}{\sigma}_{xy} \qquad (xyz) \tag{1.2}$$

where E and μ are the elastic modulus and Poisson's coefficient.

Linear equilibrium equations are valid

$$\frac{\partial \overset{\circ}{\sigma}_{xx}}{\partial x} + \frac{\partial \overset{\circ}{\sigma}_{xy}}{\partial y} + \frac{\partial \overset{\circ}{\sigma}_{xz}}{\partial z} + F^\circ{}_x = 0 \qquad (xyz) \tag{1.3}$$

with the boundary conditions

$$\overset{\circ}{\sigma}_{xx} n_x + \overset{\circ}{\sigma}_{xy} n_y + \sigma_{xz} n_z = \overset{\circ}{p}_x \qquad (xyz) \tag{1.4}$$

on that part of the surface S_1 where the external loadings $p^\circ{}_x$, $p_y{}^\circ$, $p_z{}^\circ$ are given; n_x, n_y, n_z are vector components of the external normal to the body surface in the undeformed state, and also with the boundary conditions

$$u^\circ = \bar{u}^\circ, \quad v^\circ = \bar{v}^\circ, \quad w^\circ = \bar{w}^\circ \tag{1.5}$$

on that part of the surface S_2 where the initial displacements $\bar{u}^\circ$, $\bar{v}^\circ$, $\bar{w}^\circ$ are given.

We define a new equilibrium position infinitely close to the initial one by the displacements

$$u = u^\circ + \alpha u' + \alpha^2 u'', \quad v = v^\circ + \alpha v' + \alpha^2 v'', \quad w = w^\circ + \alpha w' + \alpha^2 w'' \tag{1.6}$$

Here u', v', w' and u'', v'', w'' are considered finite functions of the x, y, z coordinates, and the parameter α is an infinitesimal independent of the coordinates.

We calculate the strain components in the new deflected equilibrium position as an expansion in the parameter α to α^2 accuracy, inclusively

$$\varepsilon_{xx} = \overset{\circ}{\varepsilon}_{xx} + \alpha\varepsilon'_{xx} + \alpha^2\varepsilon''_{xx}, \qquad \varepsilon_{xy} = \overset{\circ}{\varepsilon}_{xy} + \alpha\varepsilon'_{xy} + \alpha^2\varepsilon''_{xy} \qquad (xyz) \tag{1.7}$$

Here

$$\varepsilon'_{xx} = \frac{\partial u'}{\partial x}, \qquad \varepsilon'_{yx} = \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \qquad (xyz,\ uvw) \tag{1.8}$$

$$\varepsilon''_{xx} = \frac{\partial u''}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial x}\right)^2\right] \qquad (xyz,\ uvw)$$

$$\varepsilon''_{xy} = \frac{\partial u''}{\partial y} + \frac{\partial v''}{\partial x} + \frac{\partial u'}{\partial x}\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial x}\frac{\partial w'}{\partial y} \tag{1.9}$$

Components with factors of the type $\partial u^\circ / \partial x$, $\partial v^\circ / \partial z$ etc., which are very small as compared to unity, are omitted in the calculation of ε' and ε''. This corresponds to the assumption made earlier that the initial stress-strain state of the body is described by linear elasticity theory [2].

In the new equilibrium position the stresses are also represented as an expansion in the parameter α

$$\sigma_{xx} = \sigma_{xx}{}^\circ + \alpha\sigma'_{xx} + \alpha^2\sigma''_{xx}, \quad \sigma_{xy} = \sigma_{xy}{}^\circ + \alpha\sigma'_{xy} + \alpha^2\sigma''_{xy} \tag{1.10}$$

Following Novozhilov [2], we consider the quantities ε' and ε'' to be connected to σ' and σ'' just as σ° is connected to ε°, i.e. we consider ε' and ε'' to be expressed in terms of σ' and σ'' by utilization of dependences completely analogous to Hooke's law (1.2).

2. The total potential energy of the considered linearly elastic body loaded by potential forces, is determined by Expression

$$\begin{gathered}\vartheta=\frac{1}{2}\iiint[\sigma_{xx}\varepsilon_{xx}+\ldots+\sigma_{xy}\varepsilon_{xy}+\ldots]\,dV-\\-\iiint[F_x u+F_y v+F_z w]\,dV-\iint[p_x u+p_y v+p_z w]\,dS\end{gathered}\tag{2.1}$$

Limiting ourselves to the second degree in the parameter α, we represent the total potential energy for the deflected equilibrium position as

$$\vartheta=\vartheta^\circ+\alpha\vartheta'+\alpha^2\vartheta''\tag{2.2}$$

where the term ϑ° is the total energy of the initial unperturbed equilibrium state. The term $\alpha\vartheta'$ is the first special variation in this total energy when the possible displacements coincide with the actual displacements during bifurcation [3]. The term $\alpha^2\vartheta''$ is correspondingly proportional to the second special variation. Since the initial state is in equilibrium, the first variation in the energy is zero, and therefore $\vartheta'=0$.

Upon passage to the deflected equilibrium position, the increment in the total energy is

$$\Delta\vartheta=\vartheta-\vartheta^\circ=\alpha^2\vartheta''\tag{2.3}$$

The condition

$$\Delta\vartheta=\vartheta''=0\tag{2.4}$$

corresponds to the neutral equilibrium condition, i.e. buckling [4 and 5].

According to (1.6), (1.7), (1.10), we represent the expression for ϑ'' as

$$\vartheta''=U_1+U_2+\Pi\tag{2.5}$$

$$U_1=\frac{1}{2}\iiint\left[\sigma'_{xx}\varepsilon'_{xx}+\sigma'_{xy}\varepsilon'_{xy}+\ldots\right]dV\tag{2.6}$$

$$U_2=\frac{1}{2}\iiint[\overset{\circ}{\sigma}_{xx}\varepsilon''_{xx}+\overset{\circ}{\sigma}_{xy}\varepsilon''_{xy}+\ldots+\sigma''_{xx}\overset{\circ}{\varepsilon}_{xx}+\sigma''_{xy}\overset{\circ}{\varepsilon}_{xy}+\ldots]\,dV\tag{2.7}$$

$$\Pi=-\iiint[F_x^\circ u''+F_y^\circ v''+F_z^\circ w'']\,dV-\iint[p_x^\circ u''+p_y^\circ v''+p_z^\circ v'']\,dS\tag{2.8}$$

Using the Hooke's law (1.2), and analogous dependencies connecting σ'' and ε'', the expression for U_2 can be written thus:

$$U_2=\iiint[\overset{\circ}{\sigma}_{xx}\varepsilon''_{xx}+\overset{\circ}{\sigma}_{xy}\varepsilon''_{xy}+\ldots]\,dV\tag{2.9}$$

By using (2.9) and (1.9), components in which the displacements u'', v'', w'' enter can be extracted from the total expression (2.5) for ϑ''. Then

$$\vartheta''=A_1+A_2\tag{2.10}$$

$$\begin{gathered}A_2=\iiint\left[\overset{\circ}{\sigma}_{xx}\frac{\partial u''}{\partial x}+\overset{\circ}{\sigma}_{xy}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)+\ldots\right]dV-\\-\iiint[F_x^\circ u''+F_y^\circ v''+F_z^\circ w'']\,dV-\iint[p_x^\circ u''+p_y^\circ v''+p_z^\circ w'']\,dS\end{gathered}\tag{2.11}$$

All the remaining members are included in the symbol A_1.

It follows from (2.11) that A_2 can be considered as a new special variation in the total potential energy when the possible displacements are u'', v'', w''. Hence, $A_2=0$

for any displacements u'', v'', w'' compatible with the constraints. Taking this circumstance into account, an energy criterion for neutral equilibrium in the Bryan-Reissner form [4 and 5] can be obtained from the condition (2.4)

$$\vartheta'' = A_1 = U_1 + \iiint \Big\{ \sigma_{xx}^{\circ} \frac{1}{2}\Big[\Big(\frac{\partial u'}{\partial x}\Big)^2 + \Big(\frac{\partial v'}{\partial x}\Big)^2 + \Big(\frac{\partial w'}{\partial x}\Big)^2\Big] + \ldots$$
$$\ldots + \sigma_{xy}^{\circ}\Big[\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial x}\frac{\partial w'}{\partial y}\Big] + \ldots\Big\}\, dV \qquad (2.12)$$

The initial stresses σ° and the bifurcational displacements u', v', w' enter into this criterion, but the additional quadratic displacements u'', v'', w'' do not. In this connection it is interesting to note that Bryan himself had obtained an energy criterion of plate stability in the form (2.12) by at once putting the quadratic displacements in the plane of the plate equal to zero [4]. Not considering this deduction valid, Reissner introduced these displacements into the considerations, but in the long run he again arrived at the Bryan result. The reason for this agreement was not completely clear to Reissner; thus he writes: "It is very strange that the correct final result has been obtained in the Bryan deduction" [5].

3. To obtain an energy criterion for buckling which does not require the determination of the initial stresses and strains, we transform (2.5) differently. Since $A_2 = 0$ for arbitrary displacements u'', v'', w'' compatible with the constraints, then any additional conditions can be imposed on these displacements in transforming the expression for ϑ''; we make use of this circumstance.

According to (2.5) – (2.8), the initial stresses σ° enter into ϑ'' only in terms of (2.7) for U_2. Utilizing the accepted Hooke's law dependence between the stresses and strains, the expression for U_2 can be reduced either to (2.9), or to

$$U_2 = \iiint [\sigma_{xx}''\varepsilon_{xx}^{\circ} + \sigma_{xy}''\varepsilon_{xy}^{\circ} + \ldots]\, dV \qquad (3.1)$$

Taking account of (1.1) for ε°, we obtain

$$U_2 = \iiint \Big[\sigma_{xx}'' \frac{\partial u^\circ}{\partial x} + \sigma_{xy}''\Big(\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}\Big) + \ldots\Big]\, dV \qquad (3.2)$$

Now integrating the expression for U_2 by parts, we find

$$U_2 = -\iiint \Big[u^\circ\Big(\frac{\partial \sigma_{xx}''}{\partial x} + \frac{\partial \sigma_{xy}''}{\partial y} + \frac{\partial \sigma_{xz}''}{\partial z}\Big) + \ldots\Big]\, dV + \iint [p_x''u^\circ + p_y''v^\circ + p_z''w^\circ]\, dS \qquad (3.3)$$

$$p_x'' = \sigma_{xx}''n_x + \sigma_{yx}''n_y + \sigma_{zx}''n_z \qquad (x\, y\, z) \qquad (3.4)$$

Now let us deal with the displacements u'', v'', w'' in such a way that the triple integral, and the double integral on the part of the surface S_1, where the external loadings p_x°, p_y°, p_z° are given would vanish. To do this it is sufficient to demand compliance within the volume of the body with Eq.

$$\frac{\partial \sigma_{xx}''}{\partial x} + \frac{\partial \sigma_{xy}''}{\partial y} + \frac{\partial \sigma_{xz}''}{\partial z} = 0 \qquad (x\, y\, z) \qquad (3.5)$$

and on the part of the surface S_1 with the conditions

$$p_x'' = 0, \quad p_y'' = 0, \quad p_z'' = 0 \qquad (3.6)$$

The following boundary conditions (so that u'', v'', w'' would be compatible with the constraints)

$$u'' = 0, \quad v'' = 0, \quad w'' = 0 \tag{3.7}$$

should be satisfied on the part of the surface S_2, where the initial displacements $\bar{u}^\circ, \bar{v}^\circ, \bar{w}^\circ$ are given.

Under these conditions we obtain from (3.2)

$$U_2 = \iint_{S_2} [p_x{''}\bar{u}^\circ + p_y{''}\bar{v}^\circ + p_z{''}\bar{w}^\circ]\, dS \tag{3.8}$$

Now the energy criterion of neutral equilibrium (2.4) becomes

$$U_1 + \iint_{S_2} [p_x{''}\bar{u}^\circ + p_y{''}\bar{v}^\circ + p_z{''}\bar{w}^\circ]\, dS - \iint_{S_1} [p_x{}^\circ u'' + p_y{}^\circ v'' + p_z{}^\circ w'']\, dS -$$
$$- \iiint [F_x{}^\circ u'' + F_y{}^\circ v'' + F_z{}^\circ w'']\, dV = 0 \tag{3.9}$$

Condition (3.9) will be the desired form of the buckling energy criterion which does not contain initial stresses. The quantity U_1 is defined by (2.6), and depends only on the displacements u', v', w'. The remaining members are expressed in terms of the given loadings $p_x{}^\circ$, $p_y{}^\circ$, $p_z{}^\circ$ and $F_x{}^\circ$, $F_y{}^\circ$, $F_z{}^\circ$, and the initial displacements $\bar{u}^\circ$, $\bar{v}^\circ$, $\bar{w}^\circ$ given on the part of the surface S_2. The quadratic displacements u'', v'', w'' in condition (3.9), and the quadratic surface loads $p_x{''}$, $p_y{''}$, $p_z{''}$ are expressed in terms of the displacements u', v', w' and are also independent of the initial stresses and strains.

Those conditions which are imposed on the quadratic displacements u'', v'', w'' in order to eliminate the initial stresses from the expression for $\mathcal{E}''$ can be treated as follows. Compliance with (3.5) means that the additional quadratic displacements u'', v'', w'' have been selected such that the quadratic stresses σ'' which originate during buckling, would be self-equalized. The boundary conditions (3.6) mean that the additional quadratic loadings $p_x{''}$ $p_y{''}$ $p_z{''}$ would be zero on that part of the surface where the external surface potential loadings $p_x{}^\circ$, $p^\circ{}_y$, $p^\circ{}_z$ are given. Finally, the quadratic loadings in condition (3.9), which are on the part of the surface S_2 are additional quadratic reactions to the constraints originating during buckling.

4. The elastic stability criterion (3.9) formulated above permits the utilization of direct calculus of variations methods in solving specific problems. Taking account of the appropriate boundary conditions, we can give the displacement functions u', v', w' governing the bifurcation in the form of linear aggregates

$$u' = \sum a_i u_i{'}(x, y, z), \quad v' = \sum e_i v'(x, y, z), \quad w' = \sum c_i w_i{'}(x, y, z) \tag{4.1}$$

in the approximate solutions.

Expressing σ'' in terms of ε'', utilizing (1.9), and solving (3.5) for the quadratic displacements u'', v'', w'', we obtain a system of linear differential equations whose right sides will depend on the selected functions u', v', w'. Solving this system exactly or approximately, we find if we take account of boundary conditions (3.6) and (3.7)

$$u'' = \sum A_i u_i{''}(x, y, z), \quad v'' = \sum B_i v_i{''}(x, y, z), \quad w'' = \sum C_i w_i{''}(x, y, z) \tag{4.2}$$

where the coefficients A_i, B_i C_i depend on the parameters a_i, e_i, c_i. Now, besides the elastic system parameters, only the magnitudes of the external loadings and the coefficients a_i, e_i, c_i enter into the stability criterion (3.9); the critical values of the loadings can be determined by utilizing the known Timoshenko procedure.

The main difference between the proposed modification of the energy method and the customary method is that the determination of the initial state of stress of the system is replaced by the determination of the additional quadratic displacements u'', v'', w''. It is important that this auxiliary problem to determine u'', v'', w'' be solved independently of the specific loading of the elastic system; hence, the proposed means of solution may turn out to be simpler than the customary means in the case of the complex initial state of stress. Moreover, once having determined the displacements u'', v'', w'' for a given system under given constraint conditions, these displacements may then be utilized for other modifications of the system loading.

The customary means of solution is associated with the need to solve the problem of determining the initial state of stress each time.

Examples of utilization of the proposed modification of the energy method for stability problems of a rectangular plate loaded by concentrated forces are given in [1].

The condition $\delta(\partial'')=0$ from [2 - 4] can also be utilized in place of the neutral equilibrium condition $\Delta\partial = 0$ applied above with the Timoshenko minimization procedure. Both these methods of solution are generally equivalent [6]; following the transformations presented above, determination of the initial state of stress of an elastic system can be avoided in either.

In conclusion, let us emphasize once again that the energy stability criterion (3.9), which does not contain initial stresses, is valid when the initial stress-strain state can be described accurately enough by equations of linear elasticity theory.

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